Existence, Stability and Bifurcation of Random Complete and Periodic Solutions of Stochastic Parabolic Equations

Bixiang Wang

Department of Mathematics
New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

Email: bwang@nmt.edu

Abstract

In this paper, we study the existence, stability and bifurcation of random complete and periodic solutions for stochastic parabolic equations with multiplicative noise. We first prove the existence and uniqueness of tempered random attractors for the stochastic equations and characterize the structures of the attractors by random complete solutions. We then examine the existence and stability of random complete quasi-solutions and establish the relations of these solutions and the structures of tempered attractors. When the stochastic equations are incorporated with periodic forcing, we obtain the existence and stability of random periodic solutions. For the stochastic Chafee-Infante equation, we further establish the multiplicity and stochastic bifurcation of complete and periodic solutions.

Key words. Random attractor; periodic attractor; random periodic solution; random complete solution; bifurcation.

MSC 2010. Primary 37L55. Secondary 37L30, 35R60, 60H15.

1 Introduction

In this paper, we study the existence, stability and bifurcation of random complete and periodic solutions of a class of stochastic parabolic equations with multiplicative noise. Let Q be a bounded smooth open set in $\mathbb{R}^n$ with boundary ∂Q. Given $\tau \in \mathbb{R}$, consider the following equation on $(\tau, \infty) \times Q$:

$$\frac{\partial u}{\partial t} - \Delta u = f(t, x, u) + g(t, x) + \alpha u \circ \frac{d\omega}{dt}, \quad x \in Q \quad \text{and} \quad t > \tau, \tag{1.1}$$

with initial condition $u(\tau, \cdot) = u_\tau$ and homogeneous Drichlet boundary condition, where α is a positive constant, $g \in L^1_{loc}(\mathbb{R}, L^\infty(Q))$, ω is a two-sided real-valued Wiener process on a probability

space, and f is a locally Lipschitz continuous function satisfying some structural conditions. The stochastic equation (1.1) is understood in the sense of Stratonovich integration.

It is known that complete solutions play an important role in determining the long time dynamics of evolution equations. For instance, the structures of global attractors of some deterministic systems are completely characterized by bounded complete solutions, see, e.g., [2, 3, 13, 14, 28, 45, 48]. Similarly, the structures of random attractors of stochastic equations are fully described by random complete solutions as demonstrated in [51]. Note that random traveling wave solutions can be considered as a class of random complete solutions, and such solutions have been introduced by Shen in [46] and studied in [39, 40, 41, 42]. Since parabolic equations possess the comparison principle, the dynamical systems generated by these equations are monotone. The dynamics of monotone systems has been extensively investigated in the literature, see, e.g., [16, 17, 30, 31, 34, 43, 47]. For such systems, complete solutions can be used to describe the structures of attractors in more detail. For deterministic monotone equations, it is possible to find a stable complete solution by which the attractor is bounded from above. It is also possible to find such a solution to bound the attractor from below, see, e.g., [34, 35] and the references therein. For autonomous random systems, the role of complete solutions should be replaced by stationary solutions, and in this case, random attractors are bounded by extremal stationary solutions from above and below, respectively, see, e.g., [1, 16, 17].

Note that the articles mentioned above deal with either deterministic systems or autonomous random systems. As far as the author is aware, there is no analogous result available in the literature regarding the existence, stability and bifurcation of extremal random complete solutions for non-autonomous stochastic equations. In this paper, we will employ the non-autonomous random dynamical systems theory to investigate such solutions for parabolic equations with multiplicative noise. More precisely, we will show that the nonlinear stochastic equation (1.1) has a unique tempered random attractor in the space $C(\overline{Q})$ of continuous functions on $\overline{Q}$ with supremum norm. We will also prove that a linear equation associated with (1.1) possesses a unique tempered complete quasi-solution in $C(\overline{Q})$ which attracts all solutions. Then by the comparison principle, we show that the nonlinear equation (1.1) has two tempered complete quasi-solutions u^* and u_*, where u^* is maximal with respect to the random attractor and u_* is minimal. The stability of u^* and u_* is also obtained. As we will see later, the maximal solution u^* is stable from above and the minimal solution u_* is stable from below (see Theorem 5.1 for more details). Based on this result, we further study the bifurcation problem of random complete solutions of a specific parabolic equation, i.e., the stochastic Chafee-Infante equation. We will prove the stochastic Chafee-Infante

equation undergoes a pitchfork bifurcation when a parameter ν crosses the first eigenvalue λ_1 of the negative Laplacian with homogeneous Dirichlet boundary condition. Actually, when $\nu \le \lambda_1$, the origin is the unique random complete quasi-solution and it is stable. In this case, the random attractor of the equation is trivial. When ν passes λ_1 from below, the origin loses its stability and two more random complete quasi-solutions appear. This means the equation has at least three random complete quasi-solutions: u^*, u_* and 0 for $\nu > \lambda_1$. Furthermore, the nontrivial solutions u^* and u_* approach zero in $C(\overline{Q})$ when $\nu \to \lambda_1$ (see Theorem 6.2). Particularly, if f and g in (1.1) are periodic in time, then u^* and u_* become pathwise random periodic solutions (see Theorem 5.2). In this case, we obtain the bifurcation of pathwise random periodic solutions that seems to be the first result of its kind. The reader is referred to [23, 24, 53] for details regarding random periodic solutions.

As mentioned before, the idea of the present paper is based on the attractors theory of random dynamical systems generated by non-autonomous stochastic PDEs. The existence of random attractors for such systems has been established in [11, 22, 51]. If a stochastic equation does not contain deterministic non-autonomous forcing, then we say the equation is a autonomous stochastic one. The concept of random attractor for autonomous systems was introduced in [18, 25, 44], and the existence of such attractors have been established for various equations, see, e.g., [1, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 19, 25, 26, 27, 32, 33, 44, 49, 50]. For the existence of invariant manifolds for stochastic PDEs, we refer the reader to [20, 21, 36, 37] for details.

This paper is organized as follows. In the next section, we recall some basic concepts regarding random attractors of non-autonomous stochastic equations. In Section 3, we define a continuous random dynamical system for the stochastic equation (1.1). Section 4 is devoted to the existence of tempered random attractors and periodic attractors for (1.1). In Section 5, we discuss the structures of the attractors as well as the existence and stability of random complete quasi-solutions and random periodic solutions. We then investigate the bifurcation of random complete and periodic solutions of the Chafee-Infante equation in the last section.

Throughout this paper, we will use $C(\overline{Q})$ to denote the space of continuous functions on $\overline{Q}$ with supremum norm. We will also use $C_0(\overline{Q})$ to denote the subspace of $C(\overline{Q})$ which consists of continuous functions vanishing at ∂Q. The norm of a Banach space X is written as $\|\cdot\|_X$. The letters c and c_i ($i = 1, 2, \dots$) are generic positive constants which may change their values occasionally.

2 Preliminaries

For the reader's convenience, we recall some concepts from [51] regarding pullback attractors of non-autonomous stochastic equations. The reader is also referred to [4, 18, 19, 25, 44] for similar results on autonomous stochastic equations, and to [2, 3, 28, 45, 48] for deterministic attractors.

Hereafter, we assume that (X, d) is a complete separable metric space and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X parametrized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$. A mapping $K : \mathbb{R} \times \Omega \to 2^X$ with closed nonempty images is said to be measurable if $K(t, \cdot)$ is $(\mathcal{F},\ \mathcal{B}(X))$-measurable for every fixed $t \in \mathbb{R}$.

Definition 2.1. A mapping Φ: $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (i)-(iv) are satisfied:

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X),\ \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(iv) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

If, in addition, there exists a positive number T such that $\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot)$ for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then Φ is called a continuous periodic cocycle on X with period T.

Definition 2.2. (i) A mapping $\psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a complete orbit (solution) of Φ if for every $t \in \mathbb{R}^+$, $s, \tau \in \mathbb{R}$ and $\omega \in \Omega$, $\Phi(t, \tau + s, \theta_s\omega, \psi(s, \tau, \omega)) = \psi(t + s, \tau, \omega)$. If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t\omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then ψ is called a $\mathcal{D}$-complete orbit (solution) of Φ.

(ii) A mapping $\xi : \mathbb{R} \times \Omega \to X$ is called a complete quasi-solution of Φ if for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\Phi(t, \tau, \omega, \xi(\tau, \omega)) = \xi(\tau + t, \theta_t\omega)$. If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\xi(\tau, \omega)$ belongs to $D(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then ξ is called a $\mathcal{D}$-complete quasi-solution of Φ.

(iii) A complete quasi-solution ξ of Φ is said to be periodic with period T if $\xi(\tau + T, \omega) = \xi(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Such a solution is called a random periodic solution in [53].

Definition 2.3. A cocycle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.4. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (i)-(iii) are fulfilled:

(i) $\mathcal{A}$ is measurable and $\mathcal{A}(\tau,\omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

(ii) $\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega), \quad \forall\ t \geq 0.$$

(iii) For every $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} d(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega)) = 0.$$

If, in addition, there exists $T > 0$ such that

$$\mathcal{A}(\tau+T,\omega) = \mathcal{A}(\tau,\omega), \quad \forall\ \tau \in \mathbb{R}, \forall\ \omega \in \Omega,$$

then we say $\mathcal{A}$ is periodic with period T.

The following result can be found in [51]. For similar results, see [4, 11, 19, 25, 44].

Proposition 2.5. *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\begin{aligned}\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \bigcup_{B\in\mathcal{D}} \Omega(B,\tau,\omega)\\ = \{\psi(0,\tau,\omega) : \psi \textit{ is a } \mathcal{D}\textit{-complete solution of } \Phi\}\\ = \{\xi(\tau,\omega) : \xi \textit{ is a } \mathcal{D}\textit{-complete quasi-solution of } \Phi\}.\end{aligned}$$

If, in addition, both Φ and K are T-periodic, then so is the attractor $\mathcal{A}$.

Remark 2.6. We emphasize that the attractor $\mathcal{A}$ in Propositions 2.5 is $(\mathcal{F},\mathcal{B}(X))-$measurable which was proved in [52]. While, the measurability of $\mathcal{A}$ was only proved in [51] with respect to the P-completion of $\mathcal{F}$.

3 Nonlinear Stochastic Equations

In this section, we introduce the nonlinear stochastic PDEs we will study. Suppose Q is a bounded smooth open set in $\mathbb{R}^n$ with boundary ∂Q. Consider the stochastic parabolic equations with multiplicative noise defined on $(\tau, \infty) \times Q$ with $\tau \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} - \Delta u = f(t, x, u) + g(t, x) + \alpha u \circ \frac{d\omega}{dt}, \quad x \in Q \quad \text{and} \quad t > \tau, \tag{3.1}$$

with boundary condition

$$u = 0, \quad x \in \partial Q \quad \text{and} \quad t > \tau, \tag{3.2}$$

and initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in Q, \tag{3.3}$$

where α is a positive constant, $g \in L^1_{loc}(\mathbb{R}, L^\infty(Q))$, ω is a two-sided real-valued Wiener process on a probability space, and the symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration. The nonlinearity $f : \mathbb{R} \times \overline{Q} \times \mathbb{R} \to \mathbb{R}$ is continuous in $(t, x, s) \in \mathbb{R} \times \overline{Q} \times \mathbb{R}$. We further assume that f is locally Lipschitz continuous in $s \in \mathbb{R}$ in the sense that for any bounded intervals I_1 and I_2, there exists $L > 0$ (depending on I_1 and I_2) such that for all $t \in I_1$, $s_1, s_2 \in I_2$ and $x \in Q$,

$$|f(t, x, s_1) - f(t, x, s_2)| \le L|s_1 - s_2|. \tag{3.4}$$

Let λ be the first eigenvalue of the negative Laplacian on Q with homogeneous Dirichlet boundary condition. Suppose that there exist $\beta \in (0, \lambda)$ and $h \in L^1_{loc}(\mathbb{R}, L^\infty(Q))$ such that

$$f(t, x, 0) = 0 \quad \text{and} \quad f(t, x, s)s \le \beta s^2 + h(t, x)|s|, \qquad \text{for all} \quad t \in \mathbb{R}, \ x \in Q \quad \text{and} \quad s \in \mathbb{R}. \tag{3.5}$$

Throughout this section, we assume that δ is a fixed constant such that

$$0 < \delta < \lambda - \beta. \tag{3.6}$$

Suppose g and h satisfy the following condition: for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\delta s} \left(\|g(s, \cdot)\|_{L^\infty(Q)} + \|h(s, \cdot)\|_{L^\infty(Q)} \right) ds < \infty, \tag{3.7}$$

where δ is as in (3.6). Sometimes, we also assume g and h have the property: for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{r \to -\infty} e^{cr} \int_{-\infty}^{0} e^{\delta s} \left(\|g(s + r, \cdot)\|_{L^\infty(Q)} + \|h(s + r, \cdot)\|_{L^\infty(Q)} \right) ds = 0. \tag{3.8}$$

Note that (3.8) implies (3.7). To describe the probability space that will be used in this paper, we write $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \ \omega(0) = 0\}$. Let $\mathcal{F}$ be the Borel σ-algebra induced by the compact-open topology of Ω, and P be the corresponding Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$ given by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. In addition, there exists a θ_t-invariant set $\tilde{\Omega} \subseteq \Omega$ of full P measure such that for each $\omega \in \tilde{\Omega}$,

$$\omega(t)/t \to 0 \quad \text{as} \ \ t \to \pm\infty. \tag{3.9}$$

Hereafter, we only consider the space $\tilde{\Omega}$, and hence write $\tilde{\Omega}$ as Ω for convenience. Let $u(t, \tau, \omega, f, g, u_\tau)$ be the solution of problem (3.1)-(3.3) and v be a new variable given by

$$v(t, \tau, \omega, f, g, v_\tau) = z(t, \omega)u(t, \tau, \omega, f, g, u_\tau) \quad \text{ with } \ v_\tau = z(\tau, \omega)u_\tau, \tag{3.10}$$

where $z(t, \omega) = e^{-\alpha\omega(t)}$. From (3.1)-(3.3) we get

$$\frac{\partial v}{\partial t} - \Delta v = z(t, \omega)f(t, x, z^{-1}(t, \omega)v) + z(t, \omega)g(t, x), \quad x \in Q \quad \text{and} \quad t > \tau, \tag{3.11}$$

with

$$v|_{\partial Q} = 0 \ \text{ and } v(\tau, x) = v_\tau(x). \tag{3.12}$$

It follows from [38] that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in C_0(\overline{Q})$, there exists $T > 0$ such that the deterministic problem (3.11)-(3.12) has a unique solution $v \in C([\tau, \tau + T), C_0(\overline{Q}))$ given by the variation of constants formula

$$v(t, \tau, \omega, f, g, v_\tau) = e^{\Delta(t-\tau)}v_\tau + \int_\tau^t z(s, \omega)e^{\Delta(t-s)} \left(f(s, \cdot, z^{-1}(s, \omega)v(s)) + g(s, \cdot)\right) ds, \tag{3.13}$$

for all $t \in [\tau, \tau + T)$, where Δ is the Laplacian. Moreover, the solution depends continuously on v_τ in $C_0(\overline{Q})$ and is measurable with respect to $\omega \in \Omega$. By Lemma 3.2 below, the the solution v is actually defined for all $t > \tau$. Note that condition (3.5) implies for every $\omega \in \Omega$,

$$z(t, \omega)f(t, x, z^{-1}(t, \omega)s)s \le \beta s^2 + z(t, \omega)h(t, x)|s|, \quad t \in \mathbb{R}, \ x \in Q \ \text{ and } \ s \in \mathbb{R}. \tag{3.14}$$

We will use (3.14) to control the solutions of (3.11) by the linear equation:

$$\frac{\partial \tilde{v}}{\partial t} - \Delta\tilde{v} - \beta\tilde{v} = z(t, \omega) \left(h(t, x) + |g(t, x)|\right), \quad x \in Q \quad \text{and} \quad t > \tau, \tag{3.15}$$

with

$$\tilde{v}|_{\partial Q} = 0, \quad \text{and} \ \ \tilde{v}(\tau, x) = \tilde{v}_\tau(x), \ x \in Q. \tag{3.16}$$

For convenience, we write $A = -\Delta - \beta I$. Then it is known that A is a generator of an analytic semigroup, denoted $\{e^{-At}\}_{t\geq 0}$, in $C_0(\overline{Q})$ (see, e.g., [38]). Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\tilde{v}_\tau \in C_0(\overline{Q})$, problem (3.15)-(3.16) has a solution $\tilde{v} \in ([\tau, \infty), C_0(\overline{Q}))$ given by

$$\tilde{v}(t, \tau, \omega, g, h, \tilde{v}_\tau) = e^{-A(t-\tau)}\tilde{v}_\tau + \int_\tau^t z(s, \omega)e^{-A(t-s)} \left(h(s, \cdot) + |g(s, \cdot)|\right) ds, \tag{3.17}$$

for all $t > \tau$. A mapping $\xi : \mathbb{R} \times \Omega \to C_0(\overline{Q})$ is called a complete quasi-solution of problem (3.15)-(3.16) if for every $\tau \in \mathbb{R}$, $t > 0$ and $\omega \in \Omega$,

$$\tilde{v}(t + \tau, \tau, \omega^{-\tau}, g, h, \xi(\tau, \omega)) = \xi(\tau + t, \omega^t), \tag{3.18}$$

where ω^t is the translation of ω by t; that is, $\omega^t(\cdot) = \omega(\cdot + t)$. Note that such a mapping ξ is used in [53] for the definition of random periodic solutions. Similarly, from now on, we will use $f^t(\cdot, x, s)$, $g^t(\cdot, x)$ and $h^t(\cdot, x)$ for the translations of f, g and h in their first argument by t, respectively. The dynamics of the linear equation (3.15) is well understood as presented below.

Lemma 3.1. *Suppose* (3.7)-(3.8) *hold. Then problem* (3.15)-(3.16) *has a unique tempered complete quasi-solution* $\xi : \mathbb{R} \times \Omega \to C_0(\overline{Q})$*, which is given by, for each* $t \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$\xi(t, \omega) = \int_{-\infty}^0 e^{As} z(s, \omega) \left(h^t(s, \cdot) + |g^t(s, \cdot)|\right) ds. \tag{3.19}$$

In addition, ξ *pullback attracts all solutions in the sense that for every* $t > 0$*,* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$\begin{aligned} &\|\tilde{v}(\tau, \tau - t, \omega^{-\tau}, g, h, \tilde{v}_{\tau-t}) - \xi(\tau, \omega)\|_{C_0(\overline{Q})} \\ &\leq Me^{-(\lambda-\beta)t} \left(\|\tilde{v}_{\tau-t}\|_{C_0(\overline{Q})} + \|\xi(\tau - t, \omega^{-t})\|_{C_0(\overline{Q})}\right), \end{aligned} \tag{3.20}$$

where M *is a positive constant independent of* t*,* τ *and* ω*.*

Furthermore, if g *and* h *are periodic with period* $T > 0$*, then so is* ξ*, i.e.,* $\xi(t + T, \omega) = \xi(t, \omega)$ *for all* $t \in \mathbb{R}$ *and* $\omega \in \Omega$*.*

Proof. The proof is quite standard, see, e.g., [23, 24]. First, by (3.6) and (3.7), one can verify that the integral on the right-hand side of (3.19) is well defined. We now show that ξ is a complete quasi-solution. For convenience, we write $\varphi(t, x) = h(t, x) + |g(t, x)|$. By (3.17) and (3.19) we have

$$\begin{aligned} \tilde{v}(t, 0, \omega, g^\tau, h^\tau, \xi(\tau, \omega)) &= e^{-At}\xi(\tau, \omega) + \int_0^t e^{-A(t-s)} z(s, \omega)\varphi^\tau(s, \cdot)ds \\ &= \int_{-\infty}^t e^{-A(t-s)} z(s, \omega)\varphi^\tau(s, \cdot)ds = \int_{-\infty}^0 e^{As} z(s, \omega^t)\varphi^{t+\tau}(s, \cdot)ds. \end{aligned} \tag{3.21}$$

It follows from (3.19) and (3.21) that for each $\tau \in \mathbb{R}$, $t > 0$ and $\omega \in \Omega$,

$$v(t, 0, \omega, \varphi^{\tau}, \xi(\tau, \omega)) = \xi(\tau + t, \omega^{t}),$$

which implies (3.18) and hence ξ is a complete quasi-solution of problem (3.15)-(3.16). Next we prove that ξ is tempered. Given $c > 0$, let $\nu = \frac{1}{2}\min\{c,\ (\lambda - \beta - \delta)\}$. Note that for each $\omega \in \Omega$, there exists $s_0 < 0$ such that for all $s \leq s_0$,

$$-\alpha\omega(s) \leq -\nu s. \tag{3.22}$$

By (3.22) we have for all $t \leq s_0$, $\tau \in \mathbb{R}$ and $\omega \in \Sigma$,

$$e^{ct}\|\xi(\tau + t, \omega^{t})\|_{C_0(\overline{Q})} \leq e^{ct}\int_{-\infty}^{0} \|e^{As}z(s, \omega^{t})\varphi^{\tau+t}(s, \cdot)\|_{L^{\infty}(Q)}ds$$

$$\leq c_1 e^{ct}\int_{-\infty}^{0} e^{(\lambda-\beta)s}e^{-\alpha\omega(s+t)}\|\varphi(s + \tau + t)\|_{L^{\infty}(Q)}ds$$

$$\leq c_1 e^{(c-\nu)t}\int_{-\infty}^{0} e^{\frac{1}{2}(\lambda-\beta-\delta)s}e^{\delta s}\|\varphi(s + \tau + t)\|_{L^{\infty}(Q)}ds \leq c_1 e^{\frac{1}{2}ct}\int_{-\infty}^{0} e^{\delta s}\|\varphi(s + \tau + t)\|_{L^{\infty}(Q)}ds,$$

from which and (3.8), we get that for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{ct}\|\xi(\tau + t, \omega^{t})\|_{C_0(\overline{Q})} = 0.$$

Therefore ξ is tempered. We now establish the stability of ξ. It follows from (3.17) and (3.19) that, for every $t > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\tilde{v}(\tau, \tau - t, \omega^{-\tau}, g, h, \tilde{v}_{\tau-t}) - \xi(\tau, \omega) = \tilde{v}(t, 0, \omega^{-t}, \varphi^{\tau-t}, \tilde{v}_{\tau-t}) - \xi(\tau, \omega)$$

$$= e^{-At}\tilde{v}_{\tau-t} + \int_{0}^{t} e^{-A(t-s)}z(s, \omega^{-t})\varphi^{\tau-t}(s, \cdot)ds - \xi(\tau, \omega)$$

$$= e^{-At}\tilde{v}_{\tau-t} - \int_{-\infty}^{-t} e^{As}z(s, \omega)\varphi^{\tau}(s, \cdot)ds. \tag{3.23}$$

On the other hand, by (3.19) we have

$$e^{-At}\xi(\tau - t, \omega^{-t}) = \int_{-\infty}^{0} e^{A(s-t)}z(s, \omega^{-t})\varphi^{\tau-t}(s, \cdot)ds$$

$$= \int_{-\infty}^{-t} e^{As}z(s + t, \omega^{-t})\varphi^{\tau-t}(s + t, \cdot)ds = \int_{-\infty}^{-t} e^{As}z(s, \omega)\varphi^{\tau}(s, \cdot)ds. \tag{3.24}$$

By (3.23)-(3.24) we get that, for every $t > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\|\tilde{v}(\tau, \tau - t, \omega^{-\tau}, g, h, \tilde{v}_{\tau-t}) - \xi(\tau, \omega)\|_{C_0(\overline{Q})} = \|e^{-At}(\tilde{v}_{\tau-t} - \xi(\tau - t, \omega^{-t}))\|_{C_0(\overline{Q})}$$

$$\leq c_1 e^{-(\lambda-\beta)t}\left(\|\tilde{v}_{\tau-t}\|_{C_0(\overline{Q})} + \|\xi(\tau-t,\omega^{-t})\|_{C_0(\overline{Q})}\right). \tag{3.25}$$

Note that the uniqueness of tempered complete quasi-solutions of (3.15)-(3.16) is implied by (3.25). This completes the proof. □

Next, we establish the global existence of solutions for the deterministic problem (3.11)-(3.12).

Lemma 3.2. *Suppose* (3.4)-(3.5) *hold. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $v_\tau \in C_0(\overline{Q})$*, the solution* $v(t,\tau,\omega,f,g,v_\tau)$ *of problem* (3.11)-(3.12) *is defined for all* $t \geq \tau$.

Proof. Suppose $[\tau, T)$ be the maximal interval of existence of the solution $v(t,\tau,\omega,f,g,v_\tau)$ with $T < \infty$. We only need to prove that $v(t,\tau,\omega,f,g,v_\tau)$ is bounded in $C_0(\overline{Q})$ for all $t \in [t_0, T)$. Let $\tilde{v}(t,\tau,\omega,g,h,|v_\tau|)$ be the solution of the linear problem (3.15)-(3.16) with initial condition $|v_\tau|$. By the comparison principle we get, for all $t \geq \tau$,

$$|v(t,\tau,\omega,f,g,v_\tau)| \leq \tilde{v}(t,\tau,\omega,g,h,|v_\tau|). \tag{3.26}$$

By (3.17) we obtain, for all $t \in [\tau, T)$,

$$\|\tilde{v}(t,\tau,\omega,g,h,|v_\tau|)\|_{C_0(\overline{Q})} \leq \|e^{-A(t-\tau)}|v_\tau|\|_{C_0(\overline{Q})} + \|\int_\tau^t z(s,\omega)e^{-A(t-s)}\left(h(s,\cdot)+|g(s,\cdot)|\right)ds\|_{C_0(\overline{Q})}$$

$$\leq ce^{-(\lambda-\beta)(t-\tau)}\|v_\tau\|_{C_0(\overline{Q})} + c\int_\tau^t |z(s,\omega)|e^{-(\lambda-\beta)(t-s)}\left(\|g(s,\cdot)\|_{L^\infty(Q)} + \|h(s,\cdot)\|_{L^\infty(Q)}\right)ds.$$

Since $g,\ h \in L^1_{loc}(\mathbb{R}, L^\infty(Q))$, we find from the above that there exists $c_1 > 0$ such that

$$\|\tilde{v}(t,\tau,\omega,g,h,|v_\tau|)\|_{C_0(\overline{Q})} \leq c_1, \qquad \text{for all } \ \tau \leq t < T,$$

which along with (3.26) concludes the proof. □

By (3.10) and Lemma 3.2, we can define a map $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_0(\overline{Q}) \to C_0(\overline{Q})$ for problem (3.1)-(3.3). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in C_0(\overline{Q})$, let

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,f,g,u_\tau) = \frac{1}{z(t+\tau,\theta_{-\tau}\omega)}v(t+\tau,\tau,\theta_{-\tau}\omega,g,h,v_\tau), \tag{3.27}$$

where $v_\tau = z(\tau,\theta_{-\tau}\omega)u_\tau$. Note that v is continuous in v_τ in $C_0(\overline{Q})$ and is measurable in $\omega \in \Omega$. One can check that Φ is a continuous cocycle on $C_0(\overline{Q})$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ in the sense of Definition 2.1. By (3.10), we have the following identities which are useful in later sections, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$,

$$u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,u_{\tau-t}) = u(0,-t,\omega,f^\tau,g^\tau,u_{\tau-t})$$

$$= v(0, -t, \omega, f^\tau, g^\tau, z(-t,\omega)u_{\tau-t}) = v(\tau, \tau - t, \omega^{-\tau}, f, g, z(-t,\omega)u_{\tau-t}). \tag{3.28}$$

Let $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a tempered family of bounded nonempty subsets of $C_0(\overline{Q})$, that is, for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{r\to-\infty} e^{cr} \|D(\tau + r, \theta_r\omega)\|_{C_0(\overline{Q})} = 0, \tag{3.29}$$

where we have used the notation $\|B\|_{C_0(\overline{Q})} = \sup_{u\in B} \|u\|_{C_0(\overline{Q})}$ for a subset B of $C_0(\overline{Q})$. From now on, we use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of $C_0(\overline{Q})$, i.e.,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \ D \text{ satisfies (3.29)}\}. \tag{3.30}$$

From (3.30) we see that $\mathcal{D}$ is neighborhood closed.

4 Tempered Attractors and Periodic Attractors

In this section, we prove the existence of a unique tempered random attractor for the stochastic problem (3.1)-(3.3) with non-autonomous term g. In the case where f and g are periodic, we show the attractor is also periodic. We first derive uniform estimates of solutions in $C_0(\overline{Q})$.

Lemma 4.1. *Suppose* (3.4)-(3.8) *hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) \geq 1$ such that for all $t \geq T$ and $r \in [\tau - 1, \tau]$, the solution v of problem* (3.11)-(3.12) *satisfies*

$$\|v(r, \tau - t, \omega^{-\tau}, f, g, v_{\tau-t})\|_{C_0(\overline{Q})}$$

$$\leq M + Me^{-(\lambda-\beta)\tau} \int_{-\infty}^{\tau} e^{(\lambda-\beta)s} z(s, \omega^{-\tau}) \left(\|h(s,\cdot)\|_{L^\infty(Q)} + \|g(s,\cdot)\|_{L^\infty(Q)}\right) ds, \tag{4.1}$$

where $v_{\tau-t} = z(-t,\omega)u_{\tau-t}$ with $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, and M is a positive constant depending on λ and β, but independent of τ, ω and D.

Proof. Given $\tau \in \mathbb{R}$, $r \in [\tau - 1, \tau]$ and $t \geq 1$, by (3.17) we have

$$\|\tilde{v}(r, \tau - t, \omega^{-\tau}, g, h, v_{\tau-t})\|_{C_0(\overline{Q})} \leq \|e^{-A(r-\tau+t)} v_{\tau-t}\|_{C_0(\overline{Q})}$$

$$+ \int_{\tau-t}^{r} z(s, \omega^{-\tau}) \|e^{-A(r-s)} \left(h(s,\cdot) + |g(s,\cdot)|\right)\|_{L^\infty} ds$$

$$\leq c_1 e^{-(\lambda-\beta)(r-\tau+t)} \|v_{\tau-t}\|_{C_0(\overline{Q})}$$

$$+ c_1 e^{-(\lambda-\beta)r} \int_{\tau-t}^{r} e^{(\lambda-\beta)s} z(s,\omega^{-\tau}) \left(\|h(s,\cdot)\|_{L^\infty(Q)} + \|g(s,\cdot)\|_{L^\infty(Q)} \right). \tag{4.2}$$

Since $v_{\tau-t} = z(-t,\omega)u_{\tau-t}$ with $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$, we find that there exists $T = T(\tau,\omega,D) \geq 1$ such that for all $t \geq T$,

$$c_1 e^{-(\lambda-\beta)(r-\tau+t)} \|v_{\tau-t}\|_{C_0(\overline{Q})} \leq 1. \tag{4.3}$$

On the other hand, by (3.7) one can verify that the following integral is convergent:

$$\int_{-\infty}^{\tau} e^{(\lambda-\beta)s} z(s,\omega^{-\tau}) \left(\|h(s,\cdot)\|_{L^\infty(Q)} + \|g(s,\cdot)\|_{L^\infty(Q)} \right) ds < \infty. \tag{4.4}$$

It follows from (4.2)-(4.4) that for all $t \geq T$,

$$\|\tilde{v}(r,\tau-t,\omega^{-\tau},g,h,\tilde{v}_{\tau-t})\|_{C_0(\overline{Q})}$$

$$\leq 1 + c_1 e^{\lambda-\beta} e^{-(\lambda-\beta)\tau} \int_{-\infty}^{\tau} e^{(\lambda-\beta)s} z(s,\omega^{-\tau}) \left(\|h(s,\cdot)\|_{L^\infty(Q)} + \|g(s,\cdot)\|_{L^\infty(Q)} \right),$$

which along with the comparison principle yields the lemma. □

Based on Lemma 4.1, we have the following uniform estimates on solutions of problem (3.11)-(3.12) which imply the compactness of the solution operators.

Lemma 4.2. *Suppose* (3.4)-(3.8) *hold and* $g \in L^\infty_{loc}(\mathbb{R}, L^\infty(Q))$. *Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,D) \geq 1$ *such that for all* $t \geq T$ *and* $\gamma \in [0,1)$, *the solution* v *of problem* (3.11)-(3.12) *satisfies*

$$\|A_0^\gamma v(\tau,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})\|_{C_0(\overline{Q})} \leq C,$$

where $v_{\tau-t} = z(-t,\omega)u_{\tau-t}$ *with* $u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$, $A_0 = -\Delta$ *with homogeneous Dirichlet boundary condition, and* C *is a positive number depending on* τ *and* ω.

Proof. Note that $z(s,\omega) = e^{-\alpha\omega(s)}$ and $z^{-1}(s,\omega) = e^{\alpha\omega(s)}$ are both continuous in $s \in \mathbb{R}$. Therefore, by (3.4) and (4.1) we find that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $s \in [\tau-1,\tau]$ and $t \geq T$,

$$z(s,\omega^{-\tau}) |f\left(s,\cdot,z^{-1}(s,\omega^{-\tau})\, v\left(s,\tau-t,\omega^{-\tau},f,g,v_{\tau-t}\right)\right)| \leq c_1, \tag{4.5}$$

where $c_1 = c_1(\tau,\omega)$ is a positive number. By (3.13) we get, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 1$,

$$v(\tau,\tau-t,\omega^{-\tau},f,g,v_{\tau-t}) = v\left(\tau,\tau-1,\omega^{-\tau},f,g,v(\tau-1,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})\right)$$

$$= e^{-A_0} v(\tau-1,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})$$

$$+\int_{\tau-1}^{\tau} z(s,\omega^{-\tau})e^{-A_0(\tau-s)}\left(f\left(s,\cdot,z^{-1}(s,\omega^{-\tau})\, v\left(s,\tau-t,\omega^{-\tau},f,g,v_{\tau-t}\right)\right)+g(s,\cdot)\right)ds. \qquad (4.6)$$

It follows from (4.1), (4.5) and (4.6) that there exists $T = T(\tau,\omega,D) \geq 1$ such that for each $\gamma \in [0,1)$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq T$,

$$\|A_0^{\gamma} v(\tau,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})\|_{C_0(\overline{Q})}$$

$$\leq \|A_0^{\gamma} e^{-A_0} v(\tau-1,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})\|_{C_0(\overline{Q})}$$

$$+\int_{\tau-1}^{\tau} z(s,\omega^{-\tau})\|A_0^{\gamma}e^{-A_0(\tau-s)}\left(f\left(s,\cdot,z^{-1}(s,\omega^{-\tau})\, v\left(s,\tau-t,\omega^{-\tau},f,g,v_{\tau-t}\right)\right)+g(s,\cdot)\right)\|_{L^\infty(Q)}$$

$$\leq c_2 e^{-\lambda}\|v(\tau-1,\tau-t,\omega^{-\tau},f,g,v_{\tau-t})\|_{C_0(\overline{Q})}$$

$$+c_2\int_{\tau-1}^{\tau} e^{-\lambda(\tau-s)}\frac{z(s,\omega^{-\tau})}{(\tau-s)^{\gamma}}\left(\|f\left(s,\cdot,z^{-1}(s,\omega^{-\tau})\, v\left(s,\tau-t,\omega^{-\tau},f,g,v_{\tau-t}\right)\right)\|_{C_0(\overline{Q})}+\|g(s,\cdot)\|_{L^\infty(Q)}\right)$$

$$\leq c_3 + c_3(1+\|g\|_{L^\infty((\tau-1,\tau),L^\infty(Q))})\int_{\tau-1}^{\tau}(\tau-s)^{-\gamma}ds.$$

This completes the proof. □

Let $D(A_0^{\gamma})$ be the domain of A_0^{γ} with $\gamma > 0$. Then we know that for each $\gamma > 0$, the embedding $D(A_0^{\gamma}) \hookrightarrow C_0(\overline{Q})$ is compact. This along with Lemma 4.2 immediately implies the pullback asymptotic compactness of solutions of problem (3.11)-(3.12) in $C_0(\overline{Q})$ as stated below.

Lemma 4.3. *Suppose* (3.4)-(3.8) *hold and* $g \in L^{\infty}_{loc}(\mathbb{R}, L^{\infty}(Q))$. *Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and* $t_n \to \infty$, $v_{0,n} = z(-t_n,\omega)u_{0,n}$ *with* $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, *the sequence* $v(\tau,\tau-t_n,\omega^{-\tau},f,g,v_{0,n})$ *of solutions of problem* (3.11)-(3.12) *has a convergent subsequence in* $C_0(\overline{Q})$.

W now prove the existence of closed measurable $\mathcal{D}$-pullback absorbing sets for problem (3.1)-(3.3).

Lemma 4.4. *Suppose* (3.4)-(3.8) *hold. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.3) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *which is given by*

$$K(\tau,\omega) = \{u \in C_0(\overline{Q}) : \|u\|_{C_0(\overline{Q})} \leq R(\tau,\omega)\}$$

with

$$R(\tau,\omega) = M + Mz^{-1}(\tau,\theta_{-\tau}\omega)\int_{-\infty}^{\tau} e^{(\lambda-\beta)(s-\tau)}z(s,\theta_{-\tau}\omega)\left(\|h(s,\cdot)\|_{L^\infty(Q)}+\|g(s,\cdot)\|_{L^\infty(Q)}\right)ds,$$

where M *is a positive number depending on* λ *and* β*, but independent of* τ, ω *and* D.

Proof. By (3.28) and Lemma 4.1, there exists $T = T(\tau, \omega, D) \geq 1$ such that for all $t \geq T$,

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, f, g, u_{\tau-t})\|_{C_0(\overline{Q})} = \|v(\tau, \tau - t, \omega^{-\tau}, f, g, z(-t, \omega)u_{\tau-t})\|_{C_0(\overline{Q})}$$

$$\leq M + M \int_{-\infty}^{\tau} e^{(\lambda-\beta)(s-\tau)} z(s, \omega^{-\tau}) \left(\|h(s, \cdot)\|_{L^\infty(Q)} + \|g(s, \cdot)\|_{L^\infty(Q)}\right) ds$$

$$\leq M + M z^{-1}(\tau, \theta_{-\tau}\omega) \int_{-\infty}^{\tau} e^{(\lambda-\beta)(s-\tau)} z(s, \theta_{-\tau}\omega) \left(\|h(s, \cdot)\|_{L^\infty(Q)} + \|g(s, \cdot)\|_{L^\infty(Q)}\right) ds,$$

from which we get for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega).$$

Note that, for each $\tau \in \mathbb{R}$, $K(\tau, \cdot) : \Omega \to 2^H$ is a measurable set-valued mapping because $R(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. Next, we prove K is tempered which will complete the proof. Actually, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $r < 0$, we have

$$\|K(\tau + r, \theta_r\omega)\|_{C_0(\overline{Q})}$$

$$\leq M + M z^{-1}(\tau + r, \theta_{-\tau}\omega) \int_{-\infty}^{\tau+r} e^{(\lambda-\beta)(s-\tau-r)} z(s, \theta_{-\tau}\omega) \left(\|h(s, \cdot)\|_{L^\infty(Q)} + \|g(s, \cdot)\|_{L^\infty(Q)}\right) ds$$

$$\leq M + M e^{\alpha\omega(r)} \int_{-\infty}^{\tau+r} e^{(\lambda-\beta)(s-\tau-r)} e^{-\alpha\omega(s-\tau)} \left(\|h(s, \cdot)\|_{L^\infty(Q)} + \|g(s, \cdot)\|_{L^\infty(Q)}\right) ds$$

$$\leq M + M e^{\alpha\omega(r)} \int_{-\infty}^{0} e^{(\lambda-\beta-\delta)s} e^{-\alpha\omega(s+r)} e^{\delta s} \left(\|h(s + \tau + r, \cdot)\|_{L^\infty(Q)} + \|g(s + \tau + r, \cdot)\|_{L^\infty(Q)}\right) ds. \tag{4.7}$$

Given a positive number c, let

$$\varepsilon = \min\{\lambda - \beta - \delta, \ \frac{1}{4}c\}. \tag{4.8}$$

By (3.9) we see that there exists $N_1 < 0$ such that

$$|\alpha \ \omega(r)| \leq -\varepsilon r \quad \text{ for all } \ r \leq N_1. \tag{4.9}$$

By (4.7)-(4.9) we have, for all $r \leq N_1$,

$$\|K(\tau + r, \theta_r\omega)\|_{C_0(\overline{Q})}$$

$$\leq M + M e^{-2\varepsilon r} \int_{-\infty}^{0} e^{(\lambda-\beta-\delta-\varepsilon)s} e^{\delta s} \left(\|h(s + \tau + r, \cdot)\|_{L^\infty(Q)} + \|g(s + \tau + r, \cdot)\|_{L^\infty(Q)}\right) ds$$

$$\leq M + M e^{-\frac{1}{2}cr} \int_{-\infty}^{0} e^{\delta s} \left(\|h(s + \tau + r, \cdot)\|_{L^\infty(Q)} + \|g(s + \tau + r, \cdot)\|_{L^\infty(Q)}\right) ds. \tag{4.10}$$

By (4.10) and (3.8) we find, for every positive constant c,

$$\lim_{r\to-\infty} e^{cr}\|K(\tau+r,\theta_r\omega)\|_{C_0(\overline{Q})}=0,$$

and hence $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is tempered, as desired. $\square$

We now present the $\mathcal{D}$-pullback asymptotic compactness of problem (3.1)-(3.3).

Lemma 4.5. *Suppose* (3.4)-(3.8) *hold and* $g\in L^\infty_{loc}(\mathbb{R},L^\infty(Q))$*. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.3) *is* $\mathcal{D}$*-pullback asymptotically compact in* $C_0(\overline{Q})$*, that is, for every* $\tau\in\mathbb{R}$*,* $\omega\in\Omega$*,* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$*, and* $t_n\to\infty$*,* $u_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$*, the sequence* $\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,u_{0,n})$ *has a convergent subsequence in* $C_0(\overline{Q})$*.*

Proof. This lemma is an immediate consequence of equality (3.28) and Lemma 4.3. $\square$

As mentioned before, if Φ is $\mathcal{D}$-pullback asymptotically compact, then the attractor $\mathcal{A}$ of Φ is $(\mathcal{F},\mathcal{B}(X))-$measurable as proved in [52]. The measurability of $\mathcal{A}$ with respect to the P-completion of $\mathcal{F}$ was proved in [51]. The author is also referred to [18, 25, 44] for measurability of random attractors.

We are now ready to prove the existence of pullback attractors for problem (3.1)-(3.3).

Theorem 4.6. *Suppose* (3.4)-(3.8) *hold and* $g\in L^\infty_{loc}(\mathbb{R},L^\infty(Q))$*. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.3) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\ \omega\in\Omega\}\in\mathcal{D}$ *in* $C_0(\overline{Q})$*. Furthermore, for each* $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$*,*

$$\mathcal{A}(\tau,\omega)=\Omega(K,\tau,\omega)=\bigcup_{B\in\mathcal{D}}\Omega(B,\tau,\omega)$$

$$=\{\psi(0,\tau,\omega):\psi \text{ is a } \mathcal{D}-\text{complete solution of } \Phi\},$$

$$=\{\xi(\tau,\omega):\xi \text{ is a } \mathcal{D}-\text{complete quasi-solution of } \Phi\},$$

where K *is the closed measurable* $\mathcal{D}$*-pullback absorbing set of* Φ *given by Lemma 4.4.*

If, in addition, there exists $T>0$ *such that for all* $t\in\mathbb{R}$*,* $x\in Q$ *and* $s\in\mathbb{R}$*,*

$$f(t+T,x,s)=f(t,x,s),\ g(t+T,x)=g(t,x)\ \text{ and }\ h(t+T,x)=h(t,x), \tag{4.11}$$

then the attractor $\mathcal{A}$ *is* T*-periodic, i.e.,* $\mathcal{A}(\tau+T,\omega)=\mathcal{A}(\tau,\omega)$*.*

Proof. By Lemma 4.5 we know that Φ is $\mathcal{D}$-pullback asymptotically compact in $C_0(\overline{Q})$. Since Φ also has a closed measurable $\mathcal{D}$-pullback absorbing set K, it follows from Proposition 2.5 that Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $C_0(\overline{Q})$ with the given structure. On the other hand, if (4.11) is fulfilled, then the cocycle Φ and the absorbing set K are both T-periodic, and so is the attractor $\mathcal{A}$.

$\square$

5 Maximal and Minimal Random Complete Solutions

In this section, we first discuss the existence of tempered complete quasi-solutions of problem (3.1)-(3.3) which are maximal and minimal with respect to the random attractor $\mathcal{A}$. We then consider the existence of tempered random periodic solutions. The stability of these solutions is also examined.

Theorem 5.1. *Suppose* (3.4)-(3.8) *hold and* $g \in L^\infty_{loc}(\mathbb{R}, L^\infty(Q))$. *Then problem* (3.1)-(3.3) *has two tempered complete quasi-solutions* u^* *and* u_* *in* $C_0(\overline{Q})$ *such that for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, $u^*(\tau,\omega) \in \mathcal{A}(\tau,\omega)$, $u_*(\tau,\omega) \in \mathcal{A}(\tau,\omega)$ *and*

$$u_*(\tau,\omega)(x) \le u(x) \le u^*(\tau,\omega)(x) \quad \textit{for all} \;\; u \in \mathcal{A}(\tau,\omega) \;\; \textit{and} \;\; x \in \mathbb{R}^N, \tag{5.1}$$

where $\mathcal{A}$ *is the unique pullback attractor of problem* (3.1)-(3.3) *in* $C_0(\overline{Q})$. *The maximal complete quasi-solution* u^* *is asymptotically stable from above in the sense that for every* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and* $\psi(\tau,\omega) \in D(\tau,\omega)$ *with* $\psi(\tau,\omega) \ge u^*(\tau,\omega)$ *for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the cocycle* Φ *associated with problem* (3.1)-(3.3) *satisfies, for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{t\to\infty} \Phi(t, \tau - t, \theta_{-t}\omega, \psi(\tau - t, \theta_{-t}\omega)) = u^*(\tau,\omega), \quad \textit{in} \;\; C_0(\overline{Q}). \tag{5.2}$$

The minimal complete quasi-solution u_* *is asymptotically stable from below in the sense that for every* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and* $\psi(\tau,\omega) \in D(\tau,\omega)$ *with* $\psi(\tau,\omega) \le u_*(\tau,\omega)$ *for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the cocycle* Φ *associated with problem* (3.1)-(3.3) *satisfies, for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{t\to\infty} \Phi(t, \tau - t, \theta_{-t}\omega, \psi(\tau - t, \theta_{-t}\omega)) = u_*(\tau,\omega), \quad \textit{in} \;\; C_0(\overline{Q}). \tag{5.3}$$

Proof. Let ξ be the unique tempered complete quasi-solution of problem (3.15)-(3.16) in $C_0(\overline{Q})$ as given by (3.19). By (3.5) we find that ξ and $-\xi$ are super- and sub-solutions of problem (3.11)-(3.12), respectively. Next, we prove that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $u^*(\tau,\omega)$ and $u_*(\tau,\omega)$ in $\mathcal{A}(\tau,\omega)$ such that

$$\lim_{t\to\infty} \|u(\tau, \tau - t, \theta_{-\tau}\omega, f, g, \xi(\tau - t, \theta_{-t}\omega)) - u^*(\tau,\omega)\|_{C_0(\overline{Q})} = 0, \tag{5.4}$$

and

$$\lim_{t\to\infty} \|u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,-\xi(\tau-t,\theta_{-t}\omega))-u_*(\tau,\omega)\|_{C_0(\overline{Q})}=0, \tag{5.5}$$

where $u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\pm\xi(\tau-t,\theta_{-t}\omega))$ is the solution of problem (3.1)-(3.3) with initial data $\pm\xi(\tau-t,\theta_{-t}\omega)$ at initial time $\tau-t$. We will further prove that u^* and u_* are both tempered complete quasi-solutions and have the desired properties as stated in Theorem 5.1.

By (3.28) we find that, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\geq 0$,

$$u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega))=v(\tau,\tau-t,\omega^{-\tau},f,g,z(-t,\omega)\xi(\tau-t,\theta_{-t}\omega)). \tag{5.6}$$

By (3.19) we have

$$z(-t,\omega)\xi(\tau-t,\theta_{-t}\omega)=\xi(\tau-t,\omega^{-t}),$$

which along with (5.6) implies

$$u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega))=v(\tau,\tau-t,\omega^{-\tau},f,g,\xi(\tau-t,\omega^{-t})). \tag{5.7}$$

Let $t_1>t_2\geq 0$. By the comparison principle we have

$$v(\tau-t_2,\tau-t_1,\omega^{-\tau},f,g,\xi(\tau-t_1,\omega^{-t_1}))\leq \tilde{v}(\tau-t_2,\tau-t_1,\omega^{-\tau},g,h,\xi(\tau-t_1,\omega^{-t_1})). \tag{5.8}$$

Since ξ is a complete quasi-solution of problem (3.15)-(3.16), we get

$$\tilde{v}(\tau-t_2,\tau-t_1,\omega^{-\tau},g,h,\xi(\tau-t_1,\omega^{-t_1}))=\xi(\tau-t_2,\omega^{-t_2}),$$

which together with (5.8) shows that

$$v(\tau-t_2,\tau-t_1,\omega^{-\tau},f,g,\xi(\tau-t_1,\omega^{-t_1}))\leq\xi(\tau-t_2,\omega^{-t_2}). \tag{5.9}$$

By (5.9) and the comparison principle, we obtain

$$\begin{aligned}&v(\tau,\tau-t_2,\omega^{-\tau},f,g,\ v(\tau-t_2,\tau-t_1,\omega^{-\tau},f,g,\xi(\tau-t_1,\omega^{-t_1})))\\&\qquad\leq v(\tau,\tau-t_2,\omega^{-\tau},f,g,\ \xi(\tau-t_2,\omega^{-t_2})),\end{aligned}$$

which implies that for all $t_1>t_2\geq 0$,

$$v(\tau,\tau-t_1,\omega^{-\tau},f,g,\xi(\tau-t_1,\omega^{-t_1}))\leq v(\tau,\tau-t_2,\omega^{-\tau},f,g,\ \xi(\tau-t_2,\omega^{-t_2})). \tag{5.10}$$

Therefore, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$, $v(\tau,\tau-t,\omega^{-\tau},f,g,\xi(\tau-t,\omega^{-t}))$ is monotone in $t\in\mathbb{R}^+$, and so is $u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega))$ by (5.7). Since ξ is tempered, by the attracting property of $\mathcal{A}$, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{t\to\infty}\operatorname{dist}_{C_0(\overline{Q})}\left(u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)),\ \mathcal{A}(\tau,\omega)\right)=0. \tag{5.11}$$

By the compactness of $\mathcal{A}(\tau,\omega)$ in $C_0(\overline{Q})$, we find that for each $t \geq 0$, there is $u_0(\tau,\omega,t) \in \mathcal{A}(\tau,\omega)$ such that

$$\begin{aligned} &\|u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)) - u_0(\tau,\omega,t)\|_{C_0(\overline{Q})} \\ &= \mathrm{dist}_{C_0(\overline{Q})}\left(u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)),\ \mathcal{A}(\tau,\omega)\right). \end{aligned} \tag{5.12}$$

By (5.11)-(5.12) we find

$$\lim_{t\to\infty} \|u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)) - u_0(\tau,\omega,t)\|_{C_0(\overline{Q})} = 0. \tag{5.13}$$

Since $u_0(\tau,\omega,t) \in \mathcal{A}(\tau,\omega)$ for all $t \in \mathbb{R}^+$ and $\mathcal{A}(\tau,\omega)$ is compact in $C_0(\overline{Q})$, we see that there exist $u^*(\tau,\omega) \in \mathcal{A}(\tau,\omega)$ and a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that $u_0(\tau,\omega,t_n) \to u^*(\tau,\omega)$ in $C_0(\overline{Q})$. This and (5.13) imply

$$\lim_{n\to\infty} \|u(\tau,\tau-t_n,\theta_{-\tau}\omega,f,g,\xi(\tau-t_n,\theta_{-t_n}\omega)) - u^*(\tau,\omega)\|_{C_0(\overline{Q})} = 0. \tag{5.14}$$

Since $u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega))$ is monotone in $t \in \mathbb{R}^+$, we find from (5.14) that, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} \|u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)) - u^*(\tau,\omega)\|_{C_0(\overline{Q})} = 0. \tag{5.15}$$

Thus (5.4) follows. Given $r \in \mathbb{R}$, replacing τ by $r+\tau$ and ω by $\theta_r\omega$ in (5.15) we obtain

$$\lim_{t\to\infty} \|u(r+\tau,r+\tau-t,\theta_{-\tau}\omega,f,g,\xi(r+\tau-t,\theta_{r-t}\omega)) - u^*(r+\tau,\theta_r\omega)\|_{C_0(\overline{Q})} = 0. \tag{5.16}$$

We now prove that u^* is a complete quasi-solution of problem (3.1)-(3.3). By (5.15) and the continuity of solutions in initial data in $C_0(\overline{Q})$, we obtain that for every $r \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned} &u(r+\tau,\tau,\theta_{-\tau},f,g,u^*(\tau,\omega)) \\ &= \lim_{t\to\infty} u(r+\tau,\tau,\theta_{-\tau},f,g,u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega))) \\ &= \lim_{t\to\infty} u(r+\tau,\tau-t,\theta_{-\tau}\omega,f,g,\xi(\tau-t,\theta_{-t}\omega)) \\ &= \lim_{s\to\infty} u(r+\tau,r+\tau-s,\theta_{-\tau}\omega,f,g,\xi(r+\tau-s,\theta_{-s}\theta_r\omega)) \quad (\text{where } s = r+t) \\ &= \lim_{t\to\infty} u(r+\tau,r+\tau-t,\theta_{-\tau}\omega,f,g,\xi(r+\tau-t,\theta_{-t}\theta_r\omega)). \end{aligned} \tag{5.17}$$

It follows from (5.16)-(5.17) that for every $r \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$u(r+\tau,\tau,\theta_{-\tau}\omega,f,g,u^*(\tau,\omega)) = u^*(r+\tau,\theta_r\omega),$$

which shows that for every $r \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(r, \tau, \omega, u^*(\tau, \omega)) = u^*(r + \tau, \theta_r \omega). \tag{5.18}$$

By definition we see that u^* is a complete quasi-solution of Φ. Since $\mathcal{A}$ is tempered and $u^*(\tau, \omega) \in \mathcal{A}(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we know that u^* is also tempered.

By a similar argument, one can show that there exists a complete quasi-solution u_* such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $u_*(\tau, \omega) \in \mathcal{A}(\tau, \omega)$ and (5.5) is fulfilled.

We now prove (5.1). Let $r, s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Replacing t by r, τ by $\tau - s$ and ω by ω^{-s} in (3.20), we get

$$\begin{aligned} &\|\tilde{v}(\tau - s, \tau - s - r, \omega^{-\tau}, g, h, \tilde{v}_{\tau-s-r}) - \xi(\tau - s, \omega^{-s})\|_{C_0(\overline{Q})} \\ &\le c e^{-(\lambda-\beta)r} \left(\|\tilde{v}_{\tau-s-r}\|_{C_0(\overline{Q})} + \|\xi(\tau - s - r, \omega^{-s-r})\|_{C_0(\overline{Q})} \right). \end{aligned}$$

Letting $r = t - s$ with $t \ge s$ in the above, we obtain

$$\begin{aligned} &\|\tilde{v}(\tau - s, \tau - t, \omega^{-\tau}, g, h, \tilde{v}_{\tau-t}) - \xi(\tau - s, \omega^{-s})\|_{C_0(\overline{Q})} \\ &\le c e^{-(\lambda-\beta)(t-s)} \left(\|\tilde{v}_{\tau-t}\|_{C_0(\overline{Q})} + \|\xi(\tau - t, \omega^{-t})\|_{C_0(\overline{Q})} \right). \end{aligned} \tag{5.19}$$

Suppose $D \in \mathcal{D}$ and $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. Letting $\tilde{v}_{\tau-t} = z(-t, \omega)|u_{\tau-t}|$ in (5.19), we get

$$\begin{aligned} &\|\tilde{v}(\tau - s, \tau - t, \omega^{-\tau}, g, h, z(-t, \omega)|u_{\tau-t}|) - \xi(\tau - s, \omega^{-s})\|_{C_0(\overline{Q})} \\ &\le c e^{-(\lambda-\beta)(t-s)} \left(\|z(-t, \omega) u_{\tau-t}\|_{C_0(\overline{Q})} + \|\xi(\tau - t, \omega^{-t})\|_{C_0(\overline{Q})} \right). \end{aligned} \tag{5.20}$$

Taking the limit of (5.20) as $t \to \infty$, we obtain, for every $s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to \infty} \|\tilde{v}(\tau - s, \tau - t, \omega^{-\tau}, g, h, z(-t, \omega)|u_{\tau-t}|) - \xi(\tau - s, \omega^{-s})\|_{C_0(\overline{Q})} = 0. \tag{5.21}$$

By (5.21) and the continuity of solutions, we find that

$$\lim_{t \to \infty} v(\tau, \tau - s, \omega^{-\tau}, f, g, \tilde{v}(\tau - s, \tau - t, \omega^{-\tau}, g, h, z(-t, \omega)|u_{\tau-t}|)) = v(\tau, \tau - s, \omega^{-\tau}, f, g, \xi(\tau - s, \omega^{-s})) \tag{5.22}$$

in $C_0(\overline{Q})$. By the comparison principle, we have

$$v(\tau - s, \tau - t, \omega^{-\tau}, f, g, z(-t, \omega)|u_{\tau-t}|) \le \tilde{v}(\tau - s, \tau - t, \omega^{-\tau}, g, h, z(-t, \omega)|u_{\tau-t}|),$$

which along with the comparison principle again implies

$$v(\tau, \tau - s, \omega^{-\tau}, f, g, v(\tau - s, \tau - t, \omega^{-\tau}, f, g, z(-t, \omega) u_{\tau-t}))$$

$$\leq v(\tau,\tau-s,\omega^{-\tau},f,g,\tilde{v}(\tau-s,\tau-t,\omega^{-\tau},g,h,z(-t,\omega)|u_{\tau-t}|)).$$

In other words, we have

$$v(\tau,\tau-t,\omega^{-\tau},f,g,z(-t,\omega)u_{\tau-t}) \leq v(\tau,\tau-s,\omega^{-\tau},f,g,\tilde{v}(\tau-s,\tau-t,\omega^{-\tau},g,h,z(-t,\omega)|u_{\tau-t}|)). \tag{5.23}$$

Letting $t\to\infty$ in (5.23), by (5.22) we get

$$\limsup_{t\to\infty} v(\tau,\tau-t,\omega^{-\tau},f,g,z(-t,\omega)u_{\tau-t}) \leq v(\tau,\tau-s,\omega^{-\tau},f,g,\xi(\tau-s,\omega^{-s})). \tag{5.24}$$

It follows from (3.28), (5.7) and (5.24) that for all $s\in\mathbb{R}^+$,

$$\limsup_{t\to\infty} u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,u_{\tau-t}) \leq u(\tau,\tau-s,\theta_{-\tau}\omega,f,g,\xi(\tau-s,\theta_{-s}\omega)). \tag{5.25}$$

Letting $s\to\infty$ in (5.25), by (5.15) we get, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\limsup_{t\to\infty} u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,u_{\tau-t}) \leq u^*(\tau,\omega), \quad \text{uniformly on } \overline{Q}. \tag{5.26}$$

Given $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $u\in\mathcal{A}(\tau,\omega)$, by the invariance of $\mathcal{A}$, we find that, for every $t>0$, there is $u_{\tau-t}\in\mathcal{A}(\tau-t,\theta_{-t}\omega)$ such that $u=u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,u_{\tau-t})$. Therefore, by (5.26) we find

$$u(x)\leq u^*(\tau,\omega)(x) \quad \text{for all } x\in Q. \tag{5.27}$$

By an analogous argument, one can check that $u(x)\geq u_*(\tau,\omega)(x)$ for all $x\in Q$, and thus (5.1) follows.

We now consider the stability of u^* and u_*. Suppose $D\in\mathcal{D}$ and $\psi(\tau,\omega)\in D(\tau,\omega)$ with $\psi(\tau,\omega)\geq u^*(\tau,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$. By the comparison principle we get, for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,\psi(\tau-t,\theta_{-t}\omega)) \geq u(\tau,\tau-t,\theta_{-\tau}\omega,f,g,u^*(\tau-t,\theta_{-t}\omega)) \geq u^*(\tau,\omega),$$

which implies

$$\liminf_{t\to\infty} \Phi(t,\tau-t,\theta_{-t}\omega,\psi(\tau-t,\theta_{-t}\omega)) \geq u^*(\tau,\omega). \tag{5.28}$$

By (5.26) and (5.28) we find that

$$\lim_{t\to\infty} \Phi(t,\tau-t,\theta_{-t}\omega,\psi(\tau-t,\theta_{-t}\omega)) = u^*(\tau,\omega), \quad \text{in } C_0(\overline{Q}),$$

which gives (5.2). The convergence of (5.3) can be proved similarly and the details are omitted. □

For random periodic solutions, we have the following result.

Theorem 5.2. *Let* (3.4)-(3.5) *hold. Suppose there exists* $T > 0$ *such that* (4.11) *is valid. Then the stochastic problem* (3.1)-(3.3) *has two tempered random periodic solutions* u^* *and* u_* *which satisfy* (5.1)-(5.3).

Proof. Let u^* and u_* be the maximal and minimal complete quasi-solutions of problem (3.1)-(3.3) obtained in Theorem 5.1, respectively. By (4.11) and Lemma 3.1 we know that the unique complete quasi-solution ξ of problem (3.15)-(3.16) is periodic with period T. Then it follows from (5.15) that for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$u^*(\tau + T, \omega) = \lim_{t\to\infty} u(\tau + T, \tau + T - t, \theta_{-\tau-T}\omega, f, g, \xi(\tau + T - t, \theta_{-t}\omega))$$

$$= \lim_{t\to\infty} u(\tau, \tau - t, \theta_{-\tau}\omega, f, g, \xi(\tau - t, \theta_{-t}\omega)) = u^*(\tau, \omega).$$

This shows that u^* is T-periodic. The T-periodicity of u_* can be justified by a similar argument, and the details are omitted. □

By Theorem 5.2 we find that the random attractor of problem (3.1)-(3.3) is either trivial or it contains at least two different random periodic solutions. Based on this observation, we can prove the existence of multiple random periodic solutions when the attractor of the equation is nontrivial. This idea is demonstrated by the Chafee-Infante equation presented in the next section.

6 Bifurcation of Random Complete and Periodic Solutions

In this section, we apply the results of the previous section to a specific model called the Chafee-Infante equation, and prove the multiplicity of random complete and periodic solutions. As we will see later, these solutions undergo a pitchfork bifurcation when a parameter varies.

The one-dimensional autonomous Chafee-Infante equation is defined in $Q = (0, \pi)$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \nu u - \gamma_0 u^3, \quad x \in (0, \pi), \quad t > 0, \tag{6.1}$$

with boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad t > 0, \tag{6.2}$$

and initial condition

$$u(0, x) = u_0(x), \quad x \in (0, \pi), \tag{6.3}$$

where ν and γ_0 are positive constants.

The dynamics of problem (6.1)-(6.3) is well understood in the literature, see, e.g., [15, 29]. Let $A_0 = -\partial_{xx}$ with boundary condition (6.2). Then the eigenvalues of A_0 are $\lambda_n = n^2$ where n is any positive integer. For every $n \in \mathbb{N}$, A_0 has an eigenvector $e_n = \sin nx$ corresponding to λ_n. It is known that $\{e_n\}_{n=1}^{\infty}$ is an orthogonal basis of $L^2(Q)$. For every $\nu \in (n^2, (n+1)^2)$ with $n \in \mathbb{N}$, it was proved by Chafee and Infante in [15] that problem (6.1)-(6.3) has exactly $2n+1$ equilibrium solutions. Moreover, using a Liapunov function, one can show that problem (6.1)-(6.3) has an n-dimensional global attractor in $C_0(\overline{Q})$ which is given by the union of unstable manifolds of the $(2n+1)$ equilibrium solutions for $\nu \in (n^2, (n+1)^2)$ (see, [29]).

Suppose $\gamma : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and there exists a positive number γ_0 such that

$$\gamma(t) \geq \gamma_0, \quad \text{for all} \quad t \in \mathbb{R}. \tag{6.4}$$

We now consider the stochastically perturbed Chafee-Infante equation, for each $\tau \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \nu u - \gamma(t) u^3 + \alpha u \circ \frac{d\omega}{dt}, \quad x \in (0, \pi), \quad t > \tau, \tag{6.5}$$

with boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad t > \tau, \tag{6.6}$$

and initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in (0, \pi), \tag{6.7}$$

where α is a positive number, and ω is the two-sided real-valued Wiener process described before.

The bifurcation and the structures of attractors of equation (6.5) have been investigated in [13, 14, 34] for $\alpha = 0$ and in [12] for constant γ. We here consider the bifurcation of random complete solutions of problem (6.5)-(6.7) when γ depends on t. As a by-product, we obtain the bifurcation of random periodic solutions for periodic γ. Note that $u = 0$ is a solution of problem (6.5)-(6.7), and hence the origin is a trivial random complete solution for every $\nu \in \mathbb{R}^+$. Let $f(t, x, s) = \nu s - \gamma(t) s^3$ for all $t \in \mathbb{R}$, $x \in (0, \pi)$ and $s \in \mathbb{R}$. Recall that $\lambda_1 = 1$ is the first eigenvalue of A_0 with (6.2). By (6.4), one can check that conditions (3.4)-(3.5) are fulfilled for any $\beta \in (0, \lambda_1)$ with an appropriate positive constant function h. Then it follows from Theorem 4.6 that the stochastic problem (6.5)-(6.7) has a unique tempered pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$. It is evident that $0 \in \mathcal{A}(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \omega$. On the other hand, by Theorem 5.1 we know that problem (6.5)-(6.7) has two tempered random complete quasi-solutions u^* and u_* in $C_0(\overline{Q})$ with properties (5.1)-(5.3) and $u^*(\tau, \omega)$, $u_*(\tau, \omega) \in \mathcal{A}(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. This shows that u^* and u_* are the maximal and minimal random complete quasi-solutions, respectively. In addition, u^* is stable from above and u_* is stable from below.

Note that for each $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the solution u of problem (6.5)-(6.7) satisfies,

$$u(\tau, \tau - t, \theta_{-\tau}\omega, -u_{\tau-t}) = -u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}). \tag{6.8}$$

By (5.4)-(5.5) and (6.8) we get, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $u_*(\tau, \omega) = -u^*(\tau, \omega)$. From this and (5.1) we obtain, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$-u^*(\tau, \omega)(x) \leq u(x) \leq u^*(\tau, \omega) \quad \text{for all} \;\; u \in \mathcal{A}(\tau, \omega) \;\; \text{and} \;\; x \in \mathbb{R}^n. \tag{6.9}$$

If $\nu \in (0, \lambda_1)$, then it is clear that all solutions of problem (6.5)-(6.7) converge to zero. Therefore, the attractor $\mathcal{A}$ is trivial and $u^* = u_* = 0$. This shows that zero is the only random complete quasi-solution of problem (6.5)-(6.7) in this case.

If $\nu > \lambda_1$, then the origin becomes unstable and hence $\mathcal{A}$ is nontrivial. This along with (6.9) implies that $u^* \neq 0$. So, in this case, problem (6.5)-(6.7) has three different random complete quasi-solutions: $u = 0$, $u = u^*$ and $u = u_* = -u^*$. We will show that $\nu = \lambda_1$ is actually a bifurcation point. For that purpose, we need to prove $\mathcal{A}$ is trivial when $\nu = \lambda_1$.

Lemma 6.1. *Suppose γ is a bounded continuous function which satisfies* (6.4)*. If $\nu = \lambda_1 = 1$, then $u = 0$ is the unique tempered random complete quasi-solution of problem* (6.5)-(6.7)*. In this case, the random attractor $\mathcal{A}$ is trivial.*

Proof. By (6.9) we only need to show $u^* = 0$. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, for each $t \geq \tau$ we denote by $u(t, \tau, \omega, u^*(\tau, \theta_\tau \omega))$ the solution of problem (6.5)-(6.7) with initial condition $u^*(\tau, \theta_\tau\omega)$ at initial time τ. Since u^* is a complete quasi-solution, we find that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq \tau$,

$$u(t, \tau, \omega, u^*(\tau, \theta_\tau \omega)) = u^*(t, \theta_t \omega) \geq 0. \tag{6.10}$$

Since $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal basis of $L^2(Q)$, we may write

$$u(t, \tau, \omega, u^*(\tau, \theta_\tau\omega)) = \sum_{n=1}^{\infty} a_n(t, \tau, \omega, a_{n,\tau}) \sin nx = u_1 + u_2 \quad \text{in} \;\; L^2(Q), \tag{6.11}$$

where $u_1 = a_1(t, \tau, \omega, a_{1,\tau}) \sin x$ and $u_2 = \sum\limits_{n=2}^{\infty} a_n(t, \tau, \omega, a_{n,\tau}) \sin nx$. By (6.10)-(6.11) we have

$$a_1(t, \tau, \omega, a_{1,\tau}) = \frac{2}{\pi} \int_0^{\pi} u(t, \tau, \omega, u^*(\tau, \theta_\tau\omega)) \sin x \; dx \;\; \geq 0. \tag{6.12}$$

By (6.5) with $\nu = \lambda_1 = 1$ we get

$$\frac{da_1}{dt} = -\frac{2}{\pi}\gamma(t) \int_0^{\pi} u^3 \sin x \; dx + \alpha a_1 \circ \frac{d\omega}{dt}. \tag{6.13}$$

By Holder' inequality we have

$$\left(\int_0^\pi u\sin x\ dx\right)^3 \le 4\int_0^\pi u^3\sin x\ dx,$$

from which, (6.4) and (6.12) we get

$$\frac{2}{\pi}\gamma(t)\int_0^\pi u^3\sin x\ dx \ge \frac{\pi^2}{16}\gamma_0 a_1^3. \tag{6.14}$$

By (6.13) and (6.14) we get

$$\frac{da_1}{dt} \le -\frac{\pi^2}{16}\gamma_0 a_1^3 + \alpha a_1 \circ \frac{d\omega}{dt}. \tag{6.15}$$

Solving for a_1 from (6.15) we obtain, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\ge\tau$,

$$a_1(t,\tau,\omega,a_{1,\tau}) \le \frac{e^{\alpha(\omega(t)-\omega(\tau))}a_{1,\tau}}{\sqrt{1+\frac{\pi^2}{8}\gamma_0 a_{1,\tau}^2\int_\tau^t e^{2\alpha(\omega(s)-\omega(\tau))}ds}}. \tag{6.16}$$

It follows from (6.16) that, for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$a_1(\tau,\tau-t,\theta_{-\tau}\omega,a_{1,\tau-t}) = a_1(0,-t,\omega,a_{1,\tau-t}) \le \frac{e^{-\alpha\omega(-t)}a_{1,\tau-t}}{\sqrt{1+\frac{\pi^2}{8}\gamma_0 a_{1,\tau-t}^2\int_{-t}^0 e^{2\alpha(\omega(s)-\omega(-t))}ds}}. \tag{6.17}$$

By (6.17) we find that for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\limsup_{t\to\infty} a_1(\tau,\tau-t,\theta_{-\tau}\omega,a_{1,\tau-t}) \le \frac{2}{\pi}\left(\frac{2}{\gamma_0}\right)^{\frac12}\limsup_{t\to\infty}\left(\int_{-t}^0 e^{2\alpha\omega(s)}ds\right)^{-\frac12} = 0, \tag{6.18}$$

where we have used Lemma 2.3.41 in [1] for the last limit. Since u^* is a complete quasi-solution, by (6.12) we obtain,

$$a_1(\tau,\tau-t,\theta_{-\tau}\omega,a_{1,\tau-t}) = \frac{2}{\pi}\int_0^\pi u(\tau,\tau-t,\theta_{-\tau}\omega,u^*(\tau-t,\theta_{-t}\omega))\sin x\ dx = \frac{2}{\pi}\int_0^\pi u^*(\tau,\omega)\sin x\ dx \tag{6.19}$$

By (6.18)-(6.19) we find that for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\int_0^\pi u^*(\tau,\omega)\sin x\ dx = 0. \tag{6.20}$$

By (6.10), (6.12) and (6.20) we get, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\ge\tau$,

$$a_1(t,\tau,\omega,a_{1,\tau}) = \frac{2}{\pi}\int_0^\pi u(t,\tau,\omega,u^*(\tau,\theta_\tau\omega))\sin x\ dx = \frac{2}{\pi}\int_0^\pi u^*(t,\theta_t\omega)\sin x\ dx = 0. \tag{6.21}$$

On the other hand, by taking the inner product of (6.5) with u_2 in $L^2(Q)$, we obtain, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\ge\tau$,

$$\frac{d}{dt}\|u_2\|^2_{L^2(Q)} + 2(\lambda_2-\nu)\|u_2\|^2_{L^2(Q)} = -2\gamma(t)\int_0^\pi u^3u_2dx + 2\alpha\|u_2\|^2_{L^2(Q)}\circ\frac{d\omega}{dt}$$

$$= -2\gamma(t)\int_0^\pi u^4 dx + 2\gamma(t)\int_0^\pi u^3 u_1 dx + 2\alpha\|u_2\|^2_{L^2(Q)} \circ \frac{d\omega}{dt} \le 2\alpha\|u_2\|^2_{L^2(Q)} \circ \frac{d\omega}{dt}, \tag{6.22}$$

where the last inequality follows from (6.21). Note that $\lambda_2 = 4$ and $\nu = 1$ in the present case. Therefore, it follows from (6.22) that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \ge \tau$,

$$\|u_2(t,\tau,\omega,u_{2,\tau})\|^2_{L^2(Q)} \le e^{6(\tau-t)} e^{2\alpha(\omega(\tau)-\omega(t))} \|u_\tau\|^2_{L^2(Q)}. \tag{6.23}$$

By (6.23) we have, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\|u_2(\tau,\tau-t,\theta_{-\tau}\omega,u_2^*(\tau-t,\theta_{-t}\omega))\|^2_{L^2(Q)} \le e^{-6t} e^{-2\alpha\omega(\tau)} e^{2\alpha\omega(\tau-t)} \|u^*(\tau-t,\theta_{-t}\omega)\|^2_{L^2(Q)}. \tag{6.24}$$

By (3.9) and the temperedness of u^* in $C_0(\overline{Q})$, we obtain from (6.24) that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} \|u_2(\tau,\tau-t,\theta_{-\tau}\omega,u_2^*(\tau-t,\theta_{-t}\omega))\|^2_{L^2(Q)} = 0. \tag{6.25}$$

By (6.11), (6.21) and (6.25) we find that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} \|u(\tau,\tau-t,\theta_{-\tau}\omega,u^*(\tau-t,\theta_{-t}\omega))\|^2_{L^2(Q)} = 0. \tag{6.26}$$

Since u^* is a complete quasi-solution, from (6.26) we get $u^*(\tau,\omega) = 0$ in $L^2(Q)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. This implies $u^*(\tau,\omega)(x) = 0$ for all $x \in Q$ since $u^*(\tau,\omega) \in C_0(\overline{Q})$. $\square$

We are now ready to prove the following bifurcation result for problem (6.5)-(6.7).

Theorem 6.2. *Suppose γ is a bounded continuous function satisfying* (6.4). *Then the tempered random complete quasi-solutions of problem* (6.5)-(6.7) *undergo a pitchfork bifurcation when the parameter ν crosses $\nu = 1$ from below. More precisely, if $\nu \le 1$, problem* (6.5)-(6.7) *has a unique tempered random complete quasi-solution $u = 0$; if $\nu > 1$, the problem has three different tempered random complete quasi-solutions: u^*_ν, $-u^*_\nu$ and 0. Furthermore, $u^*_\nu(\tau,\omega) \to 0$ when $\nu \to 1$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.*

Proof. By Lemma 6.1 and what we discussed before, if $\nu \le 1$, then $u = 0$ is the only tempered random complete quasi-solution of the stochastic problem (6.5)-(6.7) and the random attractor $\mathcal{A}_\nu$ is trivial, i.e., $\mathcal{A}_\nu(\tau,\omega) = \{0\}$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. On the other hand, if $\nu > 1$ the problem has at least three tempered random complete quasi-solutions: u^*_ν, $-u^*_\nu$ and 0. The attractor $\mathcal{A}_\nu$ is nontrivial in this case. By [52], we know $\mathcal{A}_\nu$ is upper-semicontinuous when $\nu \to 1$. This means that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{\nu\to1} \mathrm{dist}_{C_0(\overline{Q})}(\mathcal{A}_\nu(\tau,\omega), \mathcal{A}_1(\tau,\omega)) = 0$. Since $u^*_\nu(\tau,\omega) \in \mathcal{A}_\nu(\tau,\omega)$ and $\mathcal{A}_1(\tau,\omega) = \{0\}$, we must have $u^*_\nu(\tau,\omega) \to 0$ as $\nu \to 1$. $\square$

Suppose now $\gamma : \mathbb{R} \to \mathbb{R}$ is T-periodic. By Theorem 5.2 we find that the quasi-solution u^*_ν above is also T-periodic. As an immediate consequence, we obtain the following bifurcation of random periodic solutions.

Corollary 6.3. *Suppose $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function satisfying* (6.4)*. Then the tempered random periodic solutions of problem* (6.5)-(6.7) *undergo a pitchfork bifurcation when the parameter ν crosses $\nu = 1$ from below. More precisely, if $\nu \le 1$, problem* (6.5)-(6.7) *has a unique tempered random periodic solution $u = 0$; if $\nu > 1$, the problem has three different tempered random periodic solutions: u^*_ν, $-u^*_\nu$ and 0. Furthermore, $u^*_\nu(\tau, \omega) \to 0$ when $\nu \to 1$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.*

References

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[2] A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[3] J.M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *J. Nonl. Sci.*, **7** (1997), 475-502.

[4] P.W. Bates, H. Lisei and K. Lu, Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, **6** (2006), 1-21.

[5] P.W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246** (2009), 845-869.

[6] T. Caraballo, J. A. Langa; On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, **10** (2003), 491-513.

[7] T. Caraballo, J. Real, I.D. Chueshov, Pullback attractors for stochastic heat equations in materials with memory, *Discrete Continuous Dynamical Systems B*, **9** (2008), 525-539.

[8] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss, and J. Valero, Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Contin. Dyn. Syst.*, **21** (2008), 415-443

[9] T. Caraballo, M.J. Garrido-Atienza, B. Schmalfuss and J. Valero, Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 439-455.

[10] T. Caraballo, M.J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.*, **74** (2011), 3671-3684.

[11] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, Pullback attractors for nonautonomous and stochastic multivalued dynamical systems, *Set-Valued Analysis*, **11** (2003), 153-201.

[12] T. Caraballo, J. A. Langa and J. C. Robinson, A stochastic pitchfork bifurcation in a Reaction-Diffusion equation, *Proc. R. Soc. Lond. A*, *457* (2001), 2041-2061.

[13] A. N. Carvalho and J. A. Langa Non-autonomous perturbation of autonomous semilinear differential equations: continuity of local stable and unstable manifolds, *J. Differential Equations*, **233** (2007), 622-653.

[14] A. N. Carvalho and J. A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, *J. Differential Equations*, **246** (2009), 2646-2668.

[15] N. Chafee and E.F. Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Applicable Analysis*, **4** (1974), 17-37.

[16] I. Chueshow, *Monotone Random Systems - Theory and Applications*, Lecture Notes in Mathematics 1779, Springer, Berlin, 2001.

[17] I. Chueshov and M. Scheutzow, On the structure of attractors and invariant measures for a class of monotone random systems, *Dynamical Systems*, **19** (2004), 127-144.

[18] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Th. Re. Fields*, **100** (1994), 365-393.

[19] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dyn. Diff. Eqns.*, **9** (1997), 307-341.

[20] J. Duan, K. Lu and B. Schmalfuss, Invariant manifolds for stochastic partial differential equations, *Ann. Probab.*, **31** (2003), 2109-2135.

[21] J. Duan, K. Lu and B. Schmalfuss, Smooth stable and unstable manifolds for stochastic evolutionary equations, *J. Dynam. Differential Equations*, **16** (2004), 949-972.

[22] J. Duan and B. Schmalfuss, The 3D quasigeostrophic fluid dynamics under random forcing on boundary, *Comm. Math. Sci.*, **1** (2003), 133-151.

[23] C. Feng and H. Zhao, Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding, *J. Functional Analysis*, **262** (2012), 4377-4422.

[24] C. Feng, H. Zhao and B. Zhou Pathwise random periodic solutions of stochastic differential equations, *J. Differential Equations*, **251** (2011), 119-149.

[25] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, **59** (1996), 21-45.

[26] M.J. Garrido-Atienza and B. Schmalfuss, Ergodicity of the infinite dimensional fractional Brownian motion, *J. Dynam. Differential Equations*, *23* (2011), 671-681.

[27] M.J. Garrido-Atienza, A. Ogrowsky and B. Schmalfuss, Random differential equations with random delays, *Stoch. Dyn.*, **11** (2011), 369-388.

[28] J.K. Hale, Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, RI, 1988.

[29] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics 840, Springer-Verlag, New York, 1981.

[30] P. Hess, *Periodic-Parabolic Boundary Value: Problems and Positivity*, Pitman, 1991.

[31] M. W. Hirsch, Stability and convergence in strongly monotone dynamical systems, *J. Reine Angew Math*, **383** (1988), 1-53.

[32] J. Huang and W. Shen, Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains, *Discrete and Continuous Dynamical Systems*, **24** (2009), 855-882.

[33] P.E. Kloeden and J.A. Langa, Flattening, squeezing and the existence of random attractors, *Proc. Royal Soc. London Serie A.*, **463** (2007), 163-181.

[34] J. A. Langa and A. Suarez, Pullback permanence for non-autonomous partial differential equations, *Electronic J. Differential Equations*, **2002** (2002), No. 72, 1-20.

[35] J. A. Langa, A. Rodriguez-Bernal and A. Suarez, On the long time behavior of non-autonomous Lotka-Volterra models with diffusion via sub-supertrajectory method, *J. Differential Equations*, **249** (2010), 414-445.

[36] Z. Lian and K. Lu, Lyapunov exponents and invariant manifolds for infinite-dimensional random dynamical systems in a Banach space, *Mem. Amer. Math. Soc.*, **206** (2010), No. 967, 1-106.

[37] S.-E. A. Mohammed, T. Zhang and H. Zhao, The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, *Mem. Amer. Math. Soc.*, **196** (2008), No. 917, 1-105.

[38] X. Mora, Semilinear parabolic problems define semiflows on C^k spaces, *Transactions of American Mathematical Society*, **278** (1983), 21-55.

[39] J. Nolen and J. Xin, A variational principle based study of KPP minimal front speeds in random shears, *Nonlinearity*, **18** (2005), 1655-1675.

[40] J. Nolen and J. Xin, A variational principle for KPP front speeds in temporally random shear flows, *Communications in Math. Phys.*, **269** (2007), 493-532.

[41] J. Nolen and J. Xin, Asymptotic spreading of KPP reactive fronts in incompressible space-time random flows, *Ann. Inst. H. Poincare Anal. Non Lineaire*, **26** (2009), 815-839.

[42] J. Nolen, An invariance principle for random traveling waves in one dimension, *SIAM J. Math. Anal.*, **43** (2011), 153-188.

[43] P. Polacik, Convergence in smooth strongly monotone flows defined by semilinear parabolic equations, *J. Differential Equations*, **79** (1989), 89-110.

[44] B. Schmalfuss, Backward cocycles and attractors of stochastic differential equations, *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, 1992, 185-192.

[45] R. Sell and Y. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.

[46] W. Shen, Traveling waves in diffusive random media, *J. Dynamics and Differential Equations*, **16** (2004), 1011-1060.

[47] H. L. Smith, *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, American Mathematical Society, Providence, 1996.

[48] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.

[49] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Transactions of American Mathematical Society*, **363** (2011), 3639-3663.

[50] B. Wang, Random Attractors for the Stochastic Benjamin-Bona-Mahony Equation on Unbounded Domains, *J. Differential Equations*, **246** (2009), 2506-2537.

[51] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012), 1544-1583.

[52] Random attractors for non-autonomous stochastic wave equations with multiplicative noise, accepted by Discrete and Continuous Dynamical Systems, Series A.

[53] H. Zhao and Z. Zheng, Random periodic solutions of random dynamical systems, *J. Differential Equations*, **246** (2009), 2020-2038.